Linear algebra at work: simulating multibody systems

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Abstract

Finding the equations of motion of a mechanical system is a difficult task for students even in simple cases - much more for the complicated three-dimensional systems studied in multibody dynamics. Using a modern simulation program this task is transferred to the software. Two simple examples will show how the equations of motion are built up systematically by using the equations of the components and simple connection rules. With enough perseverance students can do this manually. Besides basics in mechanics they need a good working knowledge of "abstract" linear algebra, which proves here to be of eminent practical use.

Multibody simulation

Simulating mechanical systems with standard programs that are based on signal processing block libraries is an interesting and basically simple task for students – as long as one knows the underlying equations of motion. To find them students usually have a rather restricted toolbox: They mainly rely on d'Alembert's principle in combination with a free body diagram. This is often not sufficient even for apparently simple examples, not to mention systems of three-dimensional solid bodies connected by joints, which are the main subject of multibody dynamics (Wittenburg (2008)).

Here modern simulation programs based on “physical modelling” come to the rescue: Their building blocks are models of simple physical systems, the connections between them are abstractions of real flanges, wires or pipes transporting physical properties in both directions. In the widely used Modelica language the components and their connections define equations, which are assembled by the simulation program to get the equations of motion automatically (Fritzson (2004)). The two examples presented in the following will demonstrate how this works in practice, even for multibody systems. This not only show the students how these programs work, but adds another method to their “equations of motion” toolbox.

Prerequisites

The physical relations that are used in the MultiBody library (basic dynamics and the Euler equation) are presented in mechanics lectures, a working skill with vector and matrix computations should result from linear algebra lessons. Usually lacking is a deeper understanding of rotation matrices. Particularly the following three relations are generally unknown to the students and have to be presented beforehand – and proven, if time admits:

- computing a rotation matrix from a fixed axis **n** and rotation angle φ

$$\boldsymbol{R} = \boldsymbol{n} \cdot \boldsymbol{n}' + (\mathbf{1} - \boldsymbol{n} \cdot \boldsymbol{n}') \cos\phi - \tilde{\boldsymbol{n}} \sin\phi$$

- writing the cross product as a matrix (i.e. hiding the Levi-Civita tensor ε)

$$\tilde{\boldsymbol{a}} := \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \Rightarrow \boldsymbol{a} \times \boldsymbol{b} = \tilde{\boldsymbol{a}} \cdot \boldsymbol{b}$$

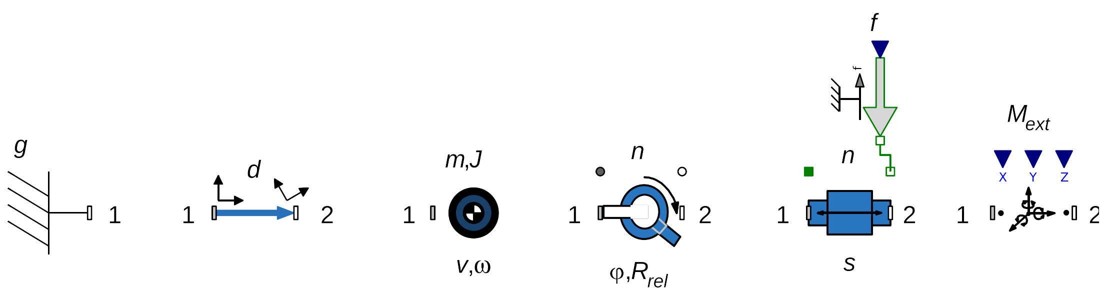
- computing the angular velocity **ω** from a time variant rotation matrix **R**(t)

$$\tilde{\omega} = \boldsymbol{R} \cdot \dot{\boldsymbol{R}}'$$

Physical modeling with Modelicas MultiBody library

Simulation software implementing Modelica usually comes with a graphical editor that allows to build models by connecting predefined building blocks. They define a set of variables and equations between them together with parameters that are fixed during the simulation. The components have connection points ("Connectors") that define the physical quantities of the block that can be accessed externally. They come in (basically) two classes: potential variables are identical at connection points and flow variables add up to zero. These relations are added to the predefined equations of each block to make up the equations of motion for the model (Fritzson (2004)).

In the Modelica MultiBody library a connector describes a local coordinate system (`frame`) relative to a globally given inertial system `world`. For this purpose it defines as potential variables the vector **r** that connects the origins of `world` and `frame` and the orthogonal matrix **R** that rotates `world` into `frame`, both defined in `world`. Corresponding flow variables are the cut force **F** and the torque **M** at the connection, both given in `frame`. In the following we will count **R** simply as three independent variables. Internally it is given by its nine matrix elements together with six constraints given by its orthogonality (Otter (2003)).



(a) **(b)** **(c)** **(d)** **(e)** **(f)**

Figure 1: MultiBody components

For the two example models we need six different components (cf. Fig. 1) and their equations, which are given directly by the functionality of a block and basic mechanical relations:

a) `World` supplies the global `world` frame and the gravity acceleration **g**

$$r_1 = 0 \qquad R_1 = \mathbf{1}$$

b) `FixedTranslation` contains two frames with a relative displacement **d**

$$\begin{aligned} r_2 &= r_1 + R_1{}'\cdot d & F_2 &= -F_1 \\ R_2 &= R_1 & M_2 &= -M_1 + d\, x\, F_1 \end{aligned}$$

c) `Body` describes a rigid body of mass m and moment of inertia **J**. Additionally it contains the internal variables **v** for the velocity and **ω** for the angular velocity.

$$\begin{aligned} v &= \dot{r}_1 & F_1 &= R_1\, m(\dot{v} - g) \\ \tilde{\omega} &= R_1 \cdot \dot{R}_1{}' & M_1 &= J\,\dot{\omega} + \omega \times (J\,\omega) \end{aligned}$$

d) `Revolute` is a joint that allows a rotation around a fixed axis **n**. Additional variables are the rotation angle φ and the relative rotation matrix $\mathbf{R}_{rel}$ between its two frames.

$$\begin{aligned} R_{rel} &= n \cdot n' + (\mathbf{1} - n \cdot n')\cos\phi - \tilde{n}\sin\phi & F_2 &= -R_{rel} \cdot F_1 \\ r_2 &= r_1 & M_2 &= -R_{rel} \cdot M_1 \\ R_2 &= R_{rel} \cdot R_1 & 0 &= M_2 \cdot n \end{aligned}$$

e) `Prismatic` permits a linear displacement of its two frames along a fixed direction **n**. The internal variable s describes the amount of the displacement. An extra input can be used to bring in an external force **f** along the axis.

$$\begin{aligned} r_2 &= r_1 + s\, R_1{}' \cdot n & F_2 &= -F_1 \\ R_2 &= R_1 & M_2 &= -M_1 + s\, n\, x\, F_1 \\ & & f &= -F_2 \cdot n \end{aligned}$$

f) `Torque` relays an externally given torque $\mathbf{M}_{ext}$ into the system.

$$\begin{aligned} F_1 &= 0 & M_1 &= M_{ext} \\ F_2 &= 0 & M_2 &= -(R_2 \cdot R_1{}') \cdot M_{ext} \end{aligned}$$

Example: robot arm

The first example is a simple model of a robot arm that is driven by an external torque, supplied e.g. by a servo motor. A corresponding Modelica model can be easily assembled using the components described above (cf. Fig. 2). The parameters of the model are fixed as

$$g = \begin{pmatrix} 0 \\ -g \\ 0 \end{pmatrix}, \quad n = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad d = \begin{pmatrix} d \\ 0 \\ 0 \end{pmatrix}, \quad M = \begin{pmatrix} 0 \\ 0 \\ M_z \end{pmatrix}, \quad J = \begin{pmatrix} J_x & 0 & 0 \\ 0 & J_z & 0 \\ 0 & 0 & J_z \end{pmatrix}$$

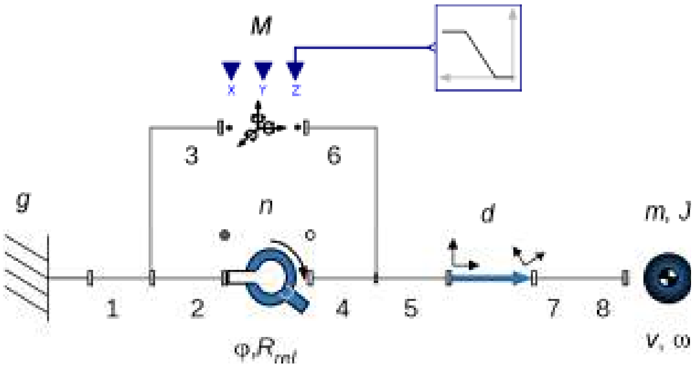


Figure 2: Model robot arm

Collecting all component and connection equations one gets the following rather huge system:

$$r_1 = 0 \quad (1)$$
$$\boldsymbol{R}_1 = \boldsymbol{1} \quad (2)$$
$$\boldsymbol{F}_3 = 0 \quad (3)$$
$$\boldsymbol{F}_6 = 0 \quad (4)$$
$$\boldsymbol{M}_3 = \boldsymbol{M} \quad (5)$$
$$\boldsymbol{M}_6 = -\boldsymbol{R}_6 \cdot \boldsymbol{R}_3' \cdot \boldsymbol{M} \quad (6)$$
$$\boldsymbol{R}_{rel} = \boldsymbol{n} \cdot \boldsymbol{n}' - \tilde{\boldsymbol{n}} \sin\phi + (\boldsymbol{1} - \boldsymbol{n} \cdot \boldsymbol{n}') \cos\phi \quad (7)$$
$$\boldsymbol{r}_2 = \boldsymbol{r}_4 \quad (8)$$
$$\boldsymbol{R}_4 = \boldsymbol{R}_{rel} \cdot \boldsymbol{R}_2 \quad (9)$$
$$\boldsymbol{F}_4 = -\boldsymbol{R}_{rel} \cdot \boldsymbol{F}_2 \quad (10)$$
$$\boldsymbol{M}_4 = -\boldsymbol{R}_{rel} \cdot \boldsymbol{M}_2 \quad (11)$$
$$0 = \boldsymbol{M}_4 \cdot \boldsymbol{n} \quad (12)$$
$$\boldsymbol{r}_7 = \boldsymbol{r}_5 + \boldsymbol{R}_5' \cdot \boldsymbol{d} \quad (13)$$
$$\boldsymbol{R}_7 = \boldsymbol{R}_5 \quad (14)$$
$$\boldsymbol{F}_7 = -\boldsymbol{F}_5 \quad (15)$$
$$\boldsymbol{M}_7 = -\boldsymbol{M}_5 + \boldsymbol{d} \times \boldsymbol{F}_5 \quad (16)$$
$$\boldsymbol{v} = \dot{\boldsymbol{r}}_8 \quad (17)$$
$$\tilde{\omega} = \boldsymbol{R}_8 \cdot \dot{\boldsymbol{R}}_8' \quad (18)$$
$$\boldsymbol{F}_8 = \boldsymbol{R}_8 \cdot m(\dot{\boldsymbol{v}} - \boldsymbol{g}) \quad (19)$$
$$\boldsymbol{M}_8 = \boldsymbol{J}\,\dot{\omega} + \omega \times \boldsymbol{J}\,\omega \quad (20)$$
$$\boldsymbol{r}_1 = \boldsymbol{r}_2 \quad (21)$$
$$\boldsymbol{r}_1 = \boldsymbol{r}_3 \quad (22)$$
$$\boldsymbol{R}_1 = \boldsymbol{R}_2 \quad (23)$$
$$\boldsymbol{R}_1 = \boldsymbol{R}_3 \quad (24)$$
$$\boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3 = 0 \quad (25)$$
$$\boldsymbol{M}_1 + \boldsymbol{M}_2 + \boldsymbol{M}_3 = 0 \quad (26)$$
$$\boldsymbol{r}_4 = \boldsymbol{r}_5 \quad (27)$$
$$\boldsymbol{r}_4 = \boldsymbol{r}_6 \quad (28)$$
$$\boldsymbol{R}_4 = \boldsymbol{R}_5 \quad (29)$$
$$\boldsymbol{R}_4 = \boldsymbol{R}_6 \quad (30)$$
$$\boldsymbol{F}_4 + \boldsymbol{F}_5 + \boldsymbol{F}_6 = 0 \quad (31)$$
$$\boldsymbol{M}_4 + \boldsymbol{M}_5 + \boldsymbol{M}_6 = 0 \quad (32)$$
$$\boldsymbol{r}_7 = \boldsymbol{r}_8 \quad (33)$$
$$\boldsymbol{R}_7 = \boldsymbol{R}_8 \quad (34)$$
$$\boldsymbol{F}_7 = -\boldsymbol{F}_8 \quad (35)$$
$$\boldsymbol{M}_7 = -\boldsymbol{M}_8 \quad (36)$$

The total number of (scalar) equations is 35x3 + 1 = 106, the number of variables is 8x4x3 + 3x3 + 1 = 106. One starts by eliminating as many variables as possible using the trivial equations and gets

$$\boldsymbol{r}_1 = 0 \quad (1) \qquad \boldsymbol{R}_4 = \boldsymbol{R}_{rel} \quad (9) \qquad \boldsymbol{F}_7 = -m\,\boldsymbol{R}_{rel}\left(\ddot{\boldsymbol{R}}_{rel}{}'\cdot\boldsymbol{d} - \boldsymbol{g}\right) \quad (35)$$
$$\boldsymbol{r}_2 = 0 \quad (21) \qquad \boldsymbol{R}_5 = \boldsymbol{R}_{rel} \quad (29) \qquad \boldsymbol{F}_6 = 0 \quad (4)$$
$$\boldsymbol{r}_3 = 0 \quad (22) \qquad \boldsymbol{R}_6 = \boldsymbol{R}_{rel} \quad (30) \qquad \boldsymbol{F}_5 = m\,\boldsymbol{R}_{rel}\left(\ddot{\boldsymbol{R}}_{rel}{}'\cdot\boldsymbol{d} - \boldsymbol{g}\right) \quad (15)$$
$$\boldsymbol{r}_4 = 0 \quad (8) \qquad \boldsymbol{R}_7 = \boldsymbol{R}_{rel} \quad (14) \qquad \boldsymbol{F}_4 = -m\,\boldsymbol{R}_{rel}\left(\ddot{\boldsymbol{R}}_{rel}{}'\cdot\boldsymbol{d} - \boldsymbol{g}\right) \quad (31)$$
$$\boldsymbol{r}_5 = 0 \quad (27) \qquad \boldsymbol{R}_8 = \boldsymbol{R}_{rel} \quad (34) \qquad \boldsymbol{F}_3 = 0 \quad (3)$$
$$\boldsymbol{r}_6 = 0 \quad (28) \qquad \boldsymbol{r}_7 = \boldsymbol{R}_{rel}{}'\cdot\boldsymbol{d} \quad (13) \qquad \boldsymbol{F}_2 = m\left(\ddot{\boldsymbol{R}}_{rel}{}'\cdot\boldsymbol{d} - \boldsymbol{g}\right) \quad (10)$$
$$\boldsymbol{R}_1 = \boldsymbol{1} \quad (2) \qquad \boldsymbol{r}_8 = \boldsymbol{R}_{rel}{}'\cdot\boldsymbol{d} \quad (33) \qquad \boldsymbol{F}_1 = -m\left(\ddot{\boldsymbol{R}}_{rel}{}'\cdot\boldsymbol{d} - \boldsymbol{g}\right) \quad (25)$$
$$\boldsymbol{R}_2 = \boldsymbol{1} \quad (23) \qquad \boldsymbol{v} = \dot{\boldsymbol{R}}_{rel}{}'\cdot\boldsymbol{d} \quad (17) \qquad \boldsymbol{M}_3 = \boldsymbol{M} \quad (5)$$
$$\boldsymbol{R}_3 = \boldsymbol{1} \quad (24) \qquad \boldsymbol{F}_8 = m\,\boldsymbol{R}_{rel}\left(\ddot{\boldsymbol{R}}_{rel}{}'\cdot\boldsymbol{d} - \boldsymbol{g}\right) \quad (19) \qquad \boldsymbol{M}_6 = -\boldsymbol{R}_{rel}\,\boldsymbol{M} \quad (6)$$

For further simplification one chooses e. g. $\mathbf{M}_8$ and $\mathbf{M}_5$ as basic and eliminates all other torques:

$$\boldsymbol{M}_7 = -\boldsymbol{M}_8 \quad (36) \qquad \boldsymbol{M}_2 = \boldsymbol{R}_{rel}{}'\cdot\boldsymbol{M}_5 - \boldsymbol{M} \quad (11)$$
$$\boldsymbol{M}_4 = -\boldsymbol{M}_5 + \boldsymbol{R}_{rel}\cdot\boldsymbol{M} \quad (32) \qquad \boldsymbol{M}_1 = -\boldsymbol{R}_{rel}{}'\cdot\boldsymbol{M}_5 \quad (26)$$

This leaves the variables $\mathbf{R}_{\mathrm{rel}}$, φ, $\boldsymbol{\omega}$, $\mathbf{M}_5$ and $\mathbf{M}_8$ together with the equations

$$\boldsymbol{R}_{rel} = \boldsymbol{n}\cdot\boldsymbol{n}' + \left(\boldsymbol{1} - \boldsymbol{n}\cdot\boldsymbol{n}'\right)\cos\phi - \tilde{\boldsymbol{n}}\sin\phi \quad (7)$$
$$\tilde{\omega} = \boldsymbol{R}_{rel}\cdot\dot{\boldsymbol{R}}_{rel}{}' \quad (18)$$
$$\boldsymbol{M}_8 = \boldsymbol{J}\,\dot{\omega} + \omega\times\boldsymbol{J}\,\omega \quad (20)$$
$$\boldsymbol{M}_5 = \boldsymbol{M}_8 + m\,\boldsymbol{d}\times\boldsymbol{R}_{rel}\left(\ddot{\boldsymbol{R}}_{rel}{}'\cdot\boldsymbol{d} - \boldsymbol{g}\right) \quad (16)$$
$$0 = \boldsymbol{n}\cdot\left(\boldsymbol{R}_{rel}\,\boldsymbol{M} - \boldsymbol{M}_5\right) \quad (12)$$

Using the explicitly given parameter values the equations are simplified substantially and one arrives at

$$\boldsymbol{R}_{rel} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \omega = \dot{\phi}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \boldsymbol{M}_8 = J_z\,\ddot{\phi}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$
$$\boldsymbol{M}_5 = \left[\left(J_z + m\,d^2\right)\ddot{\phi} + m\,d\,g\cos\phi\right]\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \left(J_z + m\,d^2\right)\ddot{\phi} + m\,d\,g\cos\phi = M_z$$

The variable φ is the angle of the joint, it is zero at horizontal position. Introducing instead the angle θ against the vertical, one finally gets the well known result

$$\left(J_z + m\,d^2\right)\ddot{\theta} + m\,d\,g\sin\theta = M_z$$

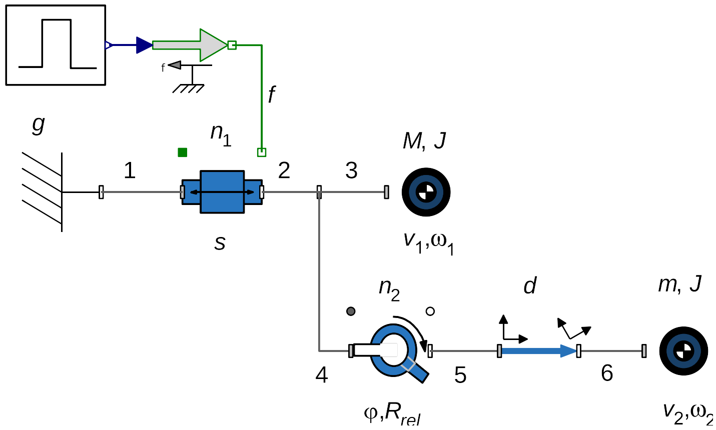


Figure 3: Model gantry crane

Example: gantry crane

The second example is a simple model of a gantry crane. Getting its equations of motion with a free body diagram is usually too difficult for the students, but a simple Modelica model can be obtained easily (cf. Fig 3). The parameters are given like before. Using the equations of the components and adding the relations defined by the connections, one gets 29x3 + 2 = 89 equations for 6x4x3 + 5x3 + 2 = 89 variables. After a simple, but tedious, computation along the lines of the first example, the equations can be reduced to

$$(m+M)\ddot{s}-(m\,d\sin\phi)\ddot{\phi}=f+m\,d\,\dot{\phi}^2\cos\phi$$
$$(-\sin\phi)\ddot{s}+d\,\ddot{\phi}=-g\cos\phi$$

By introducing the angle θ to the normal direction and isolating the second derivatives, one easily brings these equations into the standard form:

$$\ddot{s}=\frac{f+(g\cos\theta+d\,\dot{\theta}^2)\,m\sin\theta}{M+m\sin^2\theta}$$
$$\ddot{\theta}=-\frac{f\cos\theta+M\,g\sin\theta+(g+d\,\dot{\theta}^2\cos\theta)\,m\sin\theta}{d\,(M+m\sin^2\theta)}$$

If the students already know the Euler-Lagrange method, one can apply it here to obtain the same result in a shorter way. But this requires a deeper understanding of the physics, especially the formulation of the energies.

Conclusions

What is the use of such tedious computations? First of all they help to demystify the simulation software and make the “black box” translucent. Next they are a good exercise for the students who are often not used to cope with simple but long calculations and can get some practice here. Furthermore they show another method to obtain the equations of motion, which is often a difficult problem. Finally they prove that linear algebra is not just another abstract mathematical theory but has very useful applications in mechanical engineering.

References

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